



# 2

## TERMINOLOGY

contradiction  
contrapositive  
converse  
counterexample  
deductive  
equality  
equivalent  
if and only if  
implication  
inductive  
necessary  
negation  
premise  
proof  
QED  
quantifier  
reason  
reasoning  
RTP  
sufficient

## GEOMETRY

# MATHEMATICAL PROOF

- 2.01 Mathematical proof
  - 2.02 Counterexamples
  - 2.03 Converse
  - 2.04 Contrapositive statements
  - 2.05 Euclidean geometry
  - 2.06 Geometric proofs using vectors
  - 2.07 Quantifiers and Proof by Contradiction
- Chapter summary  
Chapter review




Prior learning

## THE NATURE OF PROOF

- use implication, converse, equivalence, negation, contrapositive (ACMSM024)
- use proof by contradiction (ACMSM025)
- use the symbols for implication ( $\Rightarrow$ ), equivalence ( $\Leftrightarrow$ ), and equality ( $=$ ) (ACMSM026)
- use the quantifiers 'for all' and 'there exists' (ACMSM027)
- use examples and counterexamples. (ACMSM028)

## GEOMETRIC PROOFS USING VECTORS IN THE PLANE INCLUDING

- The diagonals of a parallelogram meet at right angles if and only if it is a rhombus (ACMSM039)
- Midpoints of the sides of a quadrilateral join to form a parallelogram (ACMSM040) 

## 2.01 MATHEMATICAL PROOF

A proof in Mathematics shows something to be true for all cases. This is different from an example where something is true for a particular case. There are different types of proof: deductive, inductive and proof by contradiction. They are based on *reason*, a way of thinking attributed most often to the Classical Greek mathematicians and philosophers like Euclid, Socrates, Plato and Aristotle.

In Mathematics **deductive proof** and **proof by contradiction** are often used, but in Science **inductive proof** is common.

### IMPORTANT

A **deductive proof** is where a number of statements are made that lead logically to a conclusion. The assumptions are called **premises** or **axioms**.

An **inductive proof** is where a general conclusion is made from specific observations. In Science the general conclusion is called a **hypothesis** and it is usual to devise tests for the hypothesis.

A **proof by contradiction** assumes that the direct opposite of a statement is true and then a logical argument is used to show that the assumption is false.

## ○ Example 1

Decide whether these proofs are deductive, inductive or by contradiction.

- a All sheep are black.  
This is a sheep.  
Therefore it is black.
- b In triangle A, the angles add up to  $180^\circ$ .  
In triangle B, the angles add up to  $180^\circ$ .  
Therefore, in all triangles the angles add up to  $180^\circ$ .
- c Assume that pentagons are squares.  
Squares have 4 sides.  
Therefore pentagons have 4 sides.  
But pentagons have 5 sides, so the assumption cannot be true.  
Therefore pentagons are not squares.

### Solution

- a Proof **a** begins with some statements and a conclusion is drawn. **Proof a is deductive.**
- b Proof **b** begins with some specific statements and a general conclusion is drawn. **Proof b is inductive.**
- c Proof **c** begins with an assumption and a contradiction is shown. **Proof c is by contradiction.**

Some true statements work both ways, forwards and backwards. These are 'if and only if' or 'necessary and sufficient' statements.

Some true statements work only one way. These are 'necessary' or 'sufficient' but not both. Statements that work only one way are often written as 'If ... then ...'

## ○ Example 2

Decide whether the following statements are 'if and only if' or 'necessary but not sufficient'.

- a A square is a plane shape with 4 equal sides.
- b In a right-angled triangle with hypotenuse  $h$  and other sides  $a$  and  $b$ ,  $h^2 = a^2 + b^2$ .

### Solution

- a Check whether the statement can be true the other way: A plane shape with 4 equal sides is a square. This is not always true, but it is necessary that to be a square, it must have 4 equal sides.
- b The statement is true in reverse. If  $h^2 = a^2 + b^2$ , then the triangle must be right-angled.

For a plane shape to be a square it is necessary that it has 4 sides, but having four sides is not sufficient to make it a square, so it is necessary but not sufficient.

If a triangle is right-angled, then it is true that  $h^2 = a^2 + b^2$ . It is also true that only if three of the sides of a triangle are related by  $h^2 = a^2 + b^2$  can it be right-angled. The statement works in both directions, so it is 'if and only if' or 'necessary and sufficient'.

Arguments can be false or incorrect in different ways. It is possible to begin with correct premises and to draw a false conclusion. You can also start with incorrect premises and to draw a correct conclusion.

## ○ Example 3

Explain why each proof below is incorrect.

- a All girls have red hair.  
This is a girl.  
 $\therefore$  She has red hair.
- b Lions like to eat meat.  
Sally likes to eat meat.  
 $\therefore$  Sally is a lion.

### Solution

- a Even though the conclusion is valid, the original premise that all girls have red hair is false.
- b The two premises may be correct but Sally may not be a lion.

False premise

The conclusion is invalid because the statements are not reversible. It is not necessarily true that anything that eats meat is a lion.

## EXERCISE 2.01 Mathematical proof

### Concepts and techniques

1 **Example 1** Decide whether each proof is deductive, inductive or by contradiction.

a  $1 = 1^2$   
 $1 + 3 = 2^2$   
 $1 + 3 + 5 = 3^2$   
 $\vdots$

$\therefore 1 + 3 + 5 + \dots + (2n - 1) = n^2$

c All rectangles have diagonals that are the same length.

$ABCD$  is a rectangle.

$\therefore AC = BD$

d If  $a$  is rational and  $b$  is irrational, then  $a + b$  is irrational.

Assume that  $a + b$  is rational.

Then  $a + b$  can be written in the form  $a + b = \frac{c}{d}$ , where  $c$  and  $d$  are integers.

Since  $a$  is rational, it can also be written in the form  $a = \frac{g}{h}$ , where  $g$  and  $h$  are integers.

Clearly,  $b = (a + b) - a$

$$\begin{aligned} &= \frac{c}{d} - \frac{g}{h} \\ &= \frac{ch - gd}{dh} \end{aligned}$$

But  $(ch - gd)$  and  $dh$  are integers, so  $b$  is rational.

Therefore  $a + b$  cannot be rational, so it must be irrational.

2 Write a valid conclusion for the following deductive argument.

All quadrilaterals have an interior angle sum of  $360^\circ$ .

$PQRS$  is a kite.

A kite is a quadrilateral.

$\therefore$  \_\_\_\_\_

3 What general conclusion can be drawn from this inductive argument?

Sea temperatures rose in 2010.

Sea temperatures rose in 2011.

Sea temperatures rose in 2012.

$\therefore$  \_\_\_\_\_

4 **Example 2** Decide whether each statement is 'if and only if' or only 'necessary' but not 'sufficient'.

a All mammals are animals.

b All rhombuses have 4 equal sides.

c Congruent triangles are similar.

d All quadrilaterals with equal diagonals that bisect each other at right angles are squares.



Proof by induction



Proof by induction



Necessary and sufficient



- 5 **Example 3** Determine whether the following proofs are true or false, giving reasons.
- a A cactus is a prickly plant.  
This plant is prickly.  
 $\therefore$  It is a cactus.
- b All plane shapes with 2 pairs of adjacent sides equal are rectangles.  
WXYZ has 2 pairs of adjacent sides equal.  
 $\therefore$  WXYZ is a rectangle.
- 6 Decide what type of argument this is below (deductive or inductive) and whether or not it is valid.  
Mary has 5 children.  
All of Mary's children are boys.  
Mary is having another baby.  
 $\therefore$  It will be a boy.
- 7 Complete the proof so that it is valid.  
If a number is divisible by 4, then it is composite.  
This number is \_\_\_\_\_  
 $\therefore$  It is \_\_\_\_\_.

## Reasoning and communication

- 8 Johnny gave the following argument in a test.  
Squares have two equal diagonals.  
This shape has two equal diagonals.  
 $\therefore$  It is a square.  
Explain why this argument is invalid and alter it to make a valid argument.
- 9 Consider the following argument.  
All the people in the class have blond hair.  
All the people in the class scored well in the maths test.  
 $\therefore$  People with blond hair score well in maths tests.  
Decide whether or not the argument is valid. If not, can any valid conclusion be drawn from the statements?
- 10 Explain why this very common argument is wrong.  
Mandy didn't study for the test.  
Mandy did well in the test.  
Dennis studied for the test.  
Dennis did badly.  
 $\therefore$  Studying makes no difference so people shouldn't study for tests.

## 2.02 COUNTEREXAMPLES

A rigorous proof is required to show that a statement is true for all cases. It is much easier to show that a statement is false. All you need to do is find one case where the statement is false. This is called a **counterexample**.

## IMPORTANT

A **counterexample** is an example that shows that a statement is not true for all cases.

### ○ Example 4

Provide a counterexample to show that the following statement is false.

*All mammals have four legs.*

#### Solution

Can you think of a mammal that doesn't have four legs?

**Counterexample:**

A blue whale is a mammal, but it doesn't have four legs.

Therefore the statement is false.

### ○ Example 5

Sophie gave the following argument to her teacher.

Statement: If a number is prime, then it is odd.

Decide whether or not she is correct.

#### Solution

Are all prime numbers odd?

The number 2 is prime but it is not odd.

This is a counterexample, therefore Sophie is incorrect.

### ○ Example 6

Jack was given the statement:

*All rectangles have two pairs of opposite sides that are equal in length.*

He said that this was not true because rhombuses have two pairs of opposite sides equal in length, but a rhombus is not a rectangle. Has he provided a counterexample?

#### Solution

The statement is about rectangles. Is it possible to find a rectangle where the opposite sides are not equal?

Jack is mistaken. He has given a

counterexample to the converse statement:

All quadrilaterals with two pairs of opposite sides equal in length are rectangles.



In Example 6, Jack has assumed that the statement was ‘necessary and sufficient’ (two-way) when it was only ‘necessary’. It is necessarily true that ‘all rectangles have pairs of opposite sides equal in length’, but this is ‘not sufficient’ to show that a figure is a rectangle.

## EXERCISE 2.02 Counterexamples



Counterexamples

### Concepts and techniques

- 1 **Example 4** Find a counterexample for each statement below to demonstrate that the statement is false.
  - a If  $x^2 = 9$ , then  $x = 3$ .
  - b The statement  $x^2 - 3x + 2 = 0$  is true for  $x = 1, 2, 3, \dots$
  - c All lines that never meet are parallel.
  - d If an animal has eight legs, then it is a spider.
- 2 **Example 5** Decide whether or not the statements below are true or false and provide counterexamples for false ones.
  - a If a quadrilateral has diagonals that are equal in length, then it is a square.
  - b If  $y^2 \geq 4$ , then  $y \geq 2$ .
  - c  $a^2 + b^2 \leq (a + b)^2$  for all  $a, b \in \mathbf{R}$ .
  - d If  $nm = np$ , then  $m = p$ .
  - e All triangles with three corresponding angles equal are congruent.
- 3 **Example 6** In each case below, determine whether or not the counterexample shows that the statement is false.
  - a Statement:  $x^3 + 11x = 6x^2 + 6$  for  $x = 1, 2, 3, \dots$   
 Counterexample: For  $x = 4$ , LHS =  $4^3 + 11(4) = 108$   
 RHS =  $6(4)^2 + 6 = 102$   
 LHS  $\neq$  RHS  
 The statement is not true.
  - b Statement: All fish live in the sea.  
 Counterexample: Turtles live in the sea.  
 The statement is not true.

### Reasoning and communication

- 4 Decide whether the statement below is true or false. If it is false, find a counterexample.  
 If  $a > b$ , then  $\frac{1}{a} < \frac{1}{b}$ .
- 5 Is it always true that  $\frac{1}{n} > \frac{1}{n+1}$ ?
- 6 Is it always true that two straight lines must be either parallel or intersecting?
- 7 Are the intersections of the perpendicular bisectors of triangles always inside the triangle?
- 8 A circle can always be drawn through the three vertices of a triangle. Is this true for quadrilaterals?
- 9 Do the diagonals of a trapezium always intersect inside the trapezium?
- 10 Do the diagonals of a quadrilateral always intersect inside the quadrilateral?

## 2.03 CONVERSE

Using mathematical symbolism, we can write *if P then Q* as  $P \Rightarrow Q$ . The notation  $P \Rightarrow Q$  can also be read as *P implies Q*.

To find the converse of a statement, reverse the implication, so *if P then Q* becomes *if Q then P*, or  $P \Rightarrow Q$  becomes  $Q \Rightarrow P$ , or  $P \Leftarrow Q$ .

### IMPORTANT

For the statement *if P then Q*, the **converse** is *if Q then P*

or the **converse** of *P implies Q* is *Q implies P*.

In symbols, the **converse** of  $P \Rightarrow Q$  is  $Q \Rightarrow P$ .

The converse of a true statement may or may not be true, and vice versa. The converse may be true but the original statement may or may not be true.

### ○ Example 7

Write the converse of each statement.

- a If a cat is grey, then it is Burmese.
- b A triangle has two equal angles  $\Rightarrow$  it is isosceles.

#### Solution

- a In the original statement,  $P =$  grey cat,  
 $Q =$  Burmese cat.  
In the converse,  $Q =$  Burmese cat,  $P =$  grey cat.

If a cat is Burmese, then it is grey.

- b  $P \Rightarrow Q$  becomes  $Q \Rightarrow P$ .

An isosceles triangle  $\Rightarrow$  A triangle with two equal angles.

### ○ Example 8

Given the true statements below, write the converse of each and decide whether or not the converse is also true.

- a If  $x = 3$ , then  $x^2 = 9$ .
- b A set of points equidistant from a fixed point  $\Rightarrow$  the points are on the circumference of a circle.
- c If you are a police constable, then you wear a uniform.

#### Solution

- a We know that if  $x^2 = 9$ , then  $x = -3$  is also a solution.

Converse: If  $x^2 = 9$ , then  $x = 3$ .  
The converse is not true because  $(-3)^2 = 9$ .

b There is only one type of shape that obeys the rule.

Converse: Points on the circumference of a circle  $\Rightarrow$  A set of points equidistant from a fixed point.

The converse is true.

c Police constables are not the only people to wear a uniform.

Converse: If you wear a uniform, you are a police constable.

The converse is not true as there are other people who wear uniforms.

If a statement and its converse are both true, then this is called an **equivalence**.

## IMPORTANT

### Equivalence

If  $P \Rightarrow Q$  and  $Q \Rightarrow P$ , then we can write  $P \Leftrightarrow Q$ .

In words,  $P \Leftrightarrow Q$  means  $P$  if and only if  $Q$  or  $P$  iff  $Q$ .

### ○ Example 9

Decide whether equivalence holds for the statements below. If so, write  $A \Leftrightarrow B$ .

a A: The triangle is right-angled.

B: The sides  $a, b, c$  of the triangle are such that  $a^2 = b^2 + c^2$  ( $a$  is the longest side).

b A: You get older.

B: You have more wrinkles.

### Solution

a  $A \Rightarrow B$ : If the triangle is right-angled, then the sides  $a, b, c$  of the triangle are such that  $a^2 = b^2 + c^2$  ( $a$  is the longest side).

$B \Rightarrow A$ : If the sides  $a, b, c$  of the triangle are such that  $a^2 = b^2 + c^2$  ( $a$  is the longest side), then the triangle is right-angled.

$A \Rightarrow B$  is true.

$B \Rightarrow A$  is true.

$\therefore A \Leftrightarrow B$  or  $A$  iff  $B$ .

b  $A \Rightarrow B$ : If you get older, you have more wrinkles.

$B \Rightarrow A$ : If you have more wrinkles, you get older.

$A \Rightarrow B$  is true.

$B \Rightarrow A$  is not true as you may have had more sun exposure leading to more wrinkles, rather than age.

This is not an equivalence.

## EXERCISE 2.03 Converse

### Concepts and techniques

- 1 **Example 7** Write the converse of each statement below.
- If you live in Queensland, then you surf at the beach.
  - If you are an interesting person, then you like maths.
  - If you are the Premier, then you went to school in Darwin.
  - If an animal is a lion, then it has big teeth.
- 2 **Example 8** For each statement below, find its converse and decide whether or not it is true.
- If you do a lot of exercise, then you sweat.
  - If you study hard, then your Mathematics improves.
  - If a boy has a P-plate, then he has learnt to drive.
  - If an animal is a koala, then it lives in a tree.
- 3 **Example 9** For each true statement below, find its converse and decide whether or not it is an equivalence. If so, write it in the form  $P \text{ iff } Q$ .
- If  $x = 5$ , then  $3x = 15$ .
  - If a quadrilateral has two pairs of opposite sides equal, then it is a parallelogram.
  - If  $a > b$ , then  $a^2 > b^2$ .
  - If you have big feet, then you will have big shoes.



### Reasoning and communication

- 4 Bib and Bob were having an argument. Bib stated: If students study more maths, then they get better jobs.  
Bob stated: If students don't study more maths, then they don't get better jobs.  
Explain to Bob why his statement is not the converse of Bib's.
- 5 Shane said that the opposite of being male was being female. Sarah said this was not true because the converse of male was not female. Who was right?

## 2.04 CONTRAPOSITIVE STATEMENTS

The contrapositive statement to *if P then Q* relies on the negation of  $P$ , or *not P*, written  $P'$ ,  $\neg P$  or  $\bar{P}$ , and the negation of  $Q$ , or *not Q*, written  $Q'$ ,  $\neg Q$  or  $\bar{Q}$ .

### IMPORTANT

#### Contrapositive

If we consider the statement  $P \Rightarrow Q$ , then the contrapositive statement is  $\bar{Q} \Rightarrow \bar{P}$ .

In words, the contrapositive statement of *If P then Q* is *If not Q then not P*.

If the original statement  $P \Rightarrow Q$  is true, then the contrapositive statement  $\bar{Q} \Rightarrow \bar{P}$  is also true.

For instance, consider the statement:

*If you have been awarded a university degree, then you have studied university subjects.*

The contrapositive statement is:

*If you have not studied university subjects, then you have not been awarded a university degree.*

### ○ Example 10

Write the contrapositive of the following statements.

- a If it is raining, then the road is wet.
- b If a triangle has two equal angles, then it is an isosceles triangle.

#### Solution

- a  $P$  = it is raining,  $Q$  = the road is wet  
 $\bar{P}$  = it is not raining,  $\bar{Q}$  = the road is not wet.  
Contrapositive:  $\bar{Q} \Rightarrow \bar{P}$

The contrapositive is  
*If the road is not wet, then it is not raining.*

- b  $P$  = a triangle has two equal angles  
 $Q$  = a triangle is an isosceles triangle  
 $\bar{P}$  = a triangle does not have two equal angles  
 $\bar{Q}$  = a triangle is not an isosceles triangle  
Contrapositive:  $\bar{Q} \Rightarrow \bar{P}$

The contrapositive is  
*If it is not an isosceles triangle, then a triangle does not have two equal angles.*

You can verify that if  $P \Rightarrow Q$  is true, then the contrapositive statement  $\bar{Q} \Rightarrow \bar{P}$  is also true.

### ○ Example 11

Consider the following statement: If you are mortal, then you will die.

Identify  $P$  and  $Q$  and verify that  $P \Rightarrow Q$  is true and that the contrapositive  $\bar{Q} \Rightarrow \bar{P}$  is also true.

#### Solution

- $P$  = the first part
- $Q$  = the second part
- Is  $P \Rightarrow Q$  true?
- $\bar{P}$  = the negation of  $P$
- $\bar{Q}$  = the negation of  $Q$
- Contrapositive:  $\bar{Q} \Rightarrow \bar{P}$

$P$  = you are mortal.  
 $Q$  = you will die.  
The statement is true.  
Contrapositive:  
*If you will not die, then you are not mortal.*  
This statement is true.

Checking whether or not the contrapositive statement is true is a useful way of determining whether or not an original statement is true. A statement and its contrapositive are either both true or both false.

## ○ Example 12

Decide whether or not the statement below is true by checking the truth of the contrapositive.  
*If a quadrilateral has four right angles, then it is a square.*

### Solution

Find the contrapositive.

Is it true? No, since a rectangle is not a square but it has four right angles.

The contrapositive is:

*If it is not a square, then a quadrilateral does not have four right angles.*

The contrapositive statement is not true.

∴ The original statement is NOT true.

## INVESTIGATION

### Mathematical proof in real-life

Inverse functions rely on whether or not the converse is true for a statement  $P \Rightarrow Q$ .

For instance, if  $x = -3$ , then  $x^2 = 9$ , but the converse is not necessarily true. If  $x^2 = 9$ , then  $x = \pm 3$ . This means that the inverse function of squaring is not necessarily taking the square root, unless the original number was positive.

Security on the internet relies on secure inverse functions, along with code-breaking in general.

Find out as much as you can about the *Enigma Machine* and Alan Turing. Where is Bletchley Park?

## EXERCISE 2.04

### Contrapositive statements

#### Concepts and techniques

- Example 10** Write the contrapositive statement for each statement below.
  - If you live in Coober Pedy, then you live underground.
  - If you can drive a car, then you have a licence.
  - If you are old, then you need a hearing aid.
  - If  $x = 2$ , then  $x^2 = 4$ .
  - A cane toad is an amphibian.
  - All men are mortal.
- Given the contrapositive statements below, write down each original statement.
  - If the sky is not blue, then it is not sunny.
  - If a student does not study, then he does not pass his exams.
  - If you do not live in Alice Springs, then you do not live in the desert.
  - If a number is not positive, then it is not a counting number.
  - If a vehicle is not a car, then it does not have four wheels.
  - If you do not have a job, then you will not have money.



- 3 **Example 11** Write the contrapositive of each true statement below, then verify that its contrapositive is also true.
- If  $x \geq 3$ , then  $x^2 \geq 9$ .
  - If a quadrilateral is a rectangle, then it has diagonals equal in length.
  - People who live in Tasmania live in Australia.
  - All koalas are marsupials.
- 4 **Example 12** Write the contrapositive of each statement below, then state whether the statement and its contrapositive are both true or both false.
- If  $x \geq -3$ , then  $x^2 \geq 9$ .
  - If an animal is a bird, then it can fly.
  - If you have been to university, then you will get a good job.
  - If a number is rational, then it is real.

## Reasoning and communication

- 5 Write the contrapositive of each statement below, then determine whether the statement is true or false.
- If two triangles have matching equal angles, then they are similar shapes.
  - A quadrilateral with diagonals that meet at right angles is a rhombus.
  - All elephants have a trunk.
- 6 Mickey was trying to work out whether the following statement was true by stating the contrapositive. He decided it was true. Can you explain where he went wrong?  
 Statement: If a quadrilateral has four equal sides, then it is a square.  
 Contrapositive: If a quadrilateral does not have four equal sides, then it is not a square.
- 7 If  $P \Rightarrow Q$  is a true statement, which of the following is always true?
- A  $Q \Rightarrow P$                       B  $\neg P \Rightarrow \neg Q$                       C  $P \Leftrightarrow Q$                       D  $\neg Q \Rightarrow \neg P$

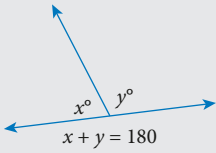
## 2.05 EUCLIDEAN GEOMETRY

In earlier years you may have studied geometry involving angles, lines, polygons, etc. This is called Euclidean geometry after the Greek mathematician Euclid (325 BC – 265 BC) who wrote about it in his now famous book *The Elements*. Euclidean geometry is the geometry of the plane, or two dimensions. There are other geometries, called *non-Euclidean geometries*, such as parabolic and hyperbolic geometries, which go beyond the plane. Euclidean geometry relies on *deductive* reasoning.

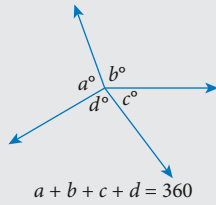
Recall the following theorems and definitions.

## IMPORTANT

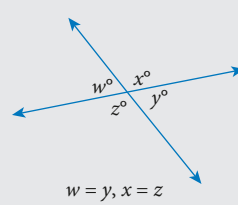
**Angles on a straight line** are supplementary (add to  $180^\circ$ )



**Angles at a point** (in a revolution) add to  $360^\circ$

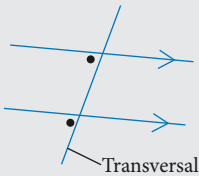


**Vertically opposite angles** are equal

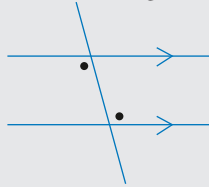


When parallel lines are crossed by a *transversal*, special pairs of angles are formed.

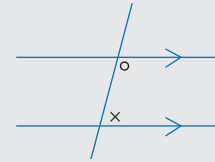
**Corresponding angles** are equal



**Alternate angles** are equal

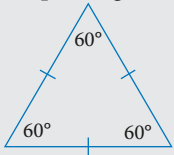


**Co-interior angles** are supplementary (add to  $180^\circ$ )



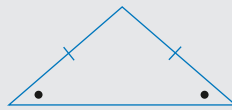
**Equilateral triangle**

3 equal sides  
3 equal angles of size  $60^\circ$



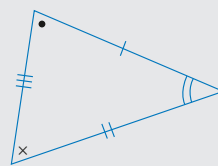
**Isosceles triangle**

2 equal sides 2 equal angles,  
opposite the equal sides

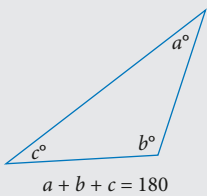


**Scalene triangle**

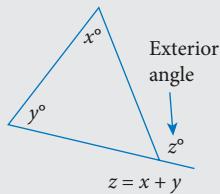
No equal sides  
No equal angles



**Angle sum of a triangle**  
Angles add up to  $180^\circ$



**Exterior angle of a triangle**  
equal to the sum of the two  
interior opposite angles



**Classifying triangles by angles**

Acute-angled triangle



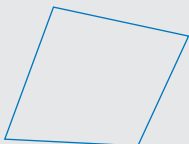
Obtuse-angled triangle



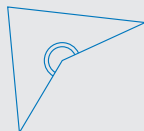
Right-angled triangle



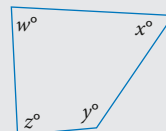
**Convex quadrilateral**



**Non-convex quadrilateral**



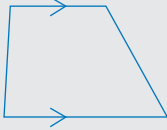
**Angle sum of a quadrilateral**



One angle greater than  $180^\circ$



### Trapezium



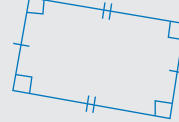
One pair of parallel sides

### Parallelogram



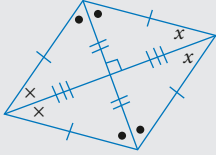
2 pairs of parallel sides,  
opposite angles equal

### Rectangle



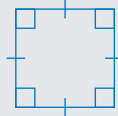
4 right angles

### Rhombus



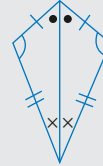
4 equal sides

### Square



4 equal sides,  
4 right angles

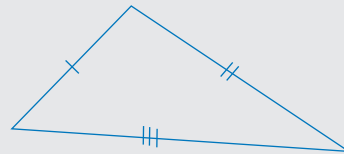
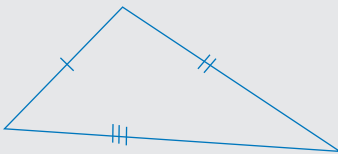
### Kite



2 pairs of equal adjacent sides

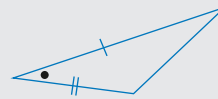
### Side, Side, Side (SSS)

If three sides are equal in both triangles, then the two triangles are congruent.



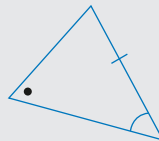
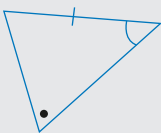
### Side, Angle, Side (SAS)

If two sides and the *included* angle are equal in both triangles, then the triangles are congruent.



### Angle, Angle, Side (AAS)

If two angles and the corresponding side are equal in both triangles, then the two triangles are congruent.



### Right angle, Hypotenuse, Side (RHS)

In a right-angled triangle, if the hypotenuse and a second side are equal in both triangles, then the two triangles are congruent.



### Similar figures

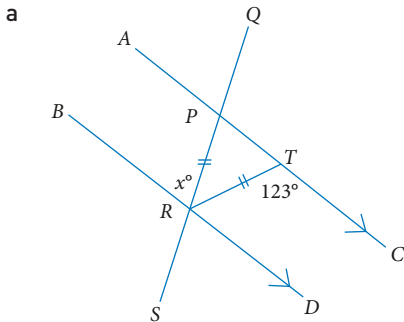
- have the same shape
- matching angles are equal
- matching sides are in the same ratio
- the symbol  $\sim$  means 'is similar to'
- vertices are named in matching order.

### Tests for similar triangles

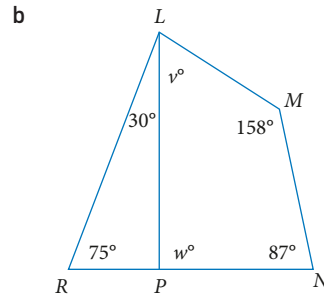
- two angles equal (equiangular)
- two sides about an included angle in proportion
- three matching sides in proportion.

### Example 13

Find the value of each pronumeral.



QS is a line.



RNML is a quadrilateral.

### Solution

- a  $\triangle PTR$  is isosceles and  $\angle CTR$  is the exterior angle of  $\triangle PTR$ . Find the base angles of  $\triangle PTR$ .

$\angle TPR$  and  $\angle PRB$  are alternate.

- b First find  $w$  using  $\triangle PRL$ .

Now find  $v$  using the quadrilateral.

$$\angle PTR = 180^\circ - 123^\circ$$

$$= 57^\circ (\angle\text{s on straight line } AC)$$

$$\therefore \angle TPR = 57^\circ (\text{base } \angle\text{s of } \triangle PTR \text{ equal})$$

$$\therefore x = 57 (\text{alternate } \angle\text{s equal, } AC \parallel BD)$$

$$w = 30 + 75$$

$$= 105 (\text{exterior } \angle \text{ of } \triangle PRL = \text{sum of two interior opposite } \angle\text{s})$$

$$v + 158 + 87 + 105 = 360$$

$$v = 10$$

$$(\angle \text{ sum of quadrilateral } PLMN)$$

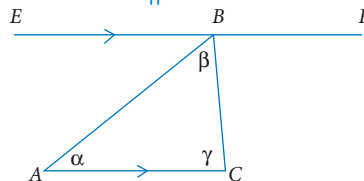
### Example 14

Prove that the angle sum of a triangle is  $180^\circ$ .

### Solution

Draw a diagram with the information you can use. Label the angles.

Construct  $\triangle ABC$  and  $EF$  passing through  $B$  such that  $EF \parallel AC$  as shown.



State what has to be proved.

RTP = required to prove

Use the given information to deduce further information, giving reasons for each step.

Use abbreviations and mathematical symbols where appropriate. Conclude the proof.

**RTP**

$$\alpha + \beta + \gamma = 180^\circ.$$

**Proof**

$\angle EBA = \alpha$  (alternate  $\angle$ s,  $EF \parallel AC$ )

$\angle FBC = \gamma$  (alternate  $\angle$ s,  $EF \parallel AC$ )

$\therefore \alpha + \beta + \gamma = 180^\circ$  ( $\angle$ s on line  $EF$ )

$\therefore$  the angle sum of a triangle is  $180^\circ$ .

## ○ Example 15

Prove that any line between two sides of a triangle that is parallel to the third side is bisected by the line from the midpoint of the third side to the opposite vertex.

### Solution

State what has to be proved.

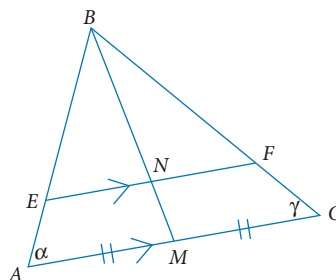
Start by drawing a diagram and naming points.

**RTP**

Any line between two sides of a triangle that is parallel to the third side is bisected by the line from the midpoint of the third side to the opposite vertex.

**Proof**

Draw a diagram with triangle  $ABC$  and choose  $E$  on  $AB$ . Draw a line through  $E$  parallel to  $AC$  and name its intersection with  $BC$  point  $F$ . Draw a line from  $B$  to  $M$ , the midpoint of  $AC$ . Label the intersection with  $EF$  as  $N$ .



To prove the bisection, you need to show that  $EN = NF$ . You will use similar triangles. Name the base angles for convenience.

First prove that  $\triangle BAM \parallel \triangle BEN$ . Use the parallel lines. Do the first angle.

Do the next angle.

Do the third angle.

State the similarity.

Name  $\angle BAC = \alpha$  and  $\angle BCA = \gamma$ .

$\angle BAM = \angle BEN$  (corresponding angles)

$\angle BMA = \angle BNE$  (corresponding angles)

$\angle ABM = \angle EBN$  (common angle)

Thus  $\triangle BAM \parallel \triangle BEN$  (equiangular)

The proof of similarity between the triangles on the other side is identical, apart from the names.

Use similarity of  $\triangle BAM$  and  $\triangle BEN$ .

Use similarity of  $\triangle BCM$  and  $\triangle BFN$ .

Use the equalities to obtain the desired result.

Use the fact that  $M$  is the midpoint.

Draw the conclusion.

Say you have done it!

Similarly,  $\triangle BCM \parallel \triangle BFN$

$$\frac{AM}{EN} = \frac{BM}{BN} \quad (\triangle BAM \parallel \triangle BEN)$$

$$\frac{CM}{FN} = \frac{BM}{BN} \quad (\triangle BCM \parallel \triangle BFN)$$

$$\text{Thus } \frac{AM}{EN} = \frac{CM}{FN} \quad (\text{both equal to } \frac{BM}{BN}).$$

$$\text{But } AM = CM$$

$$\text{Thus } EN = FN$$

**QED**

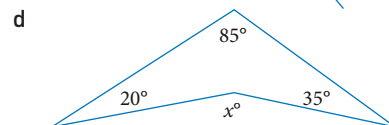
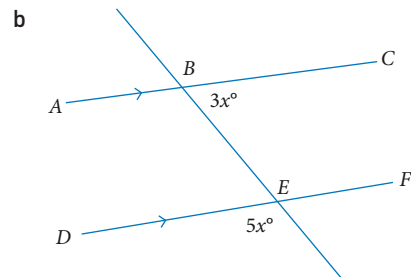
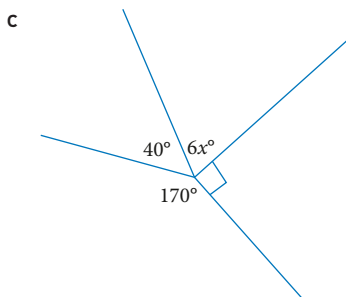
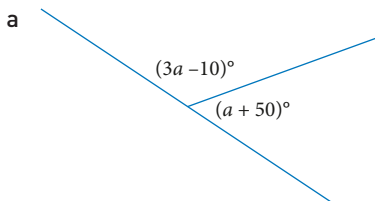
Example 15 proves that any line parallel to the side of a triangle will be bisected by the corresponding median. This result is a further deduction arising from the axiom that parallel lines never meet (Playfair's axiom).

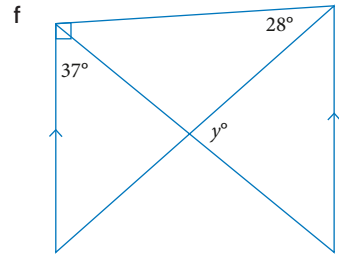
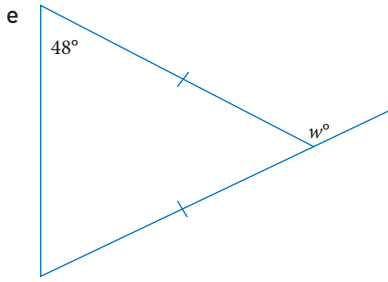
Congruence proofs and similarity proofs are also part of Euclidean geometry.

## EXERCISE 2.05 Euclidean geometry

### Concepts and techniques

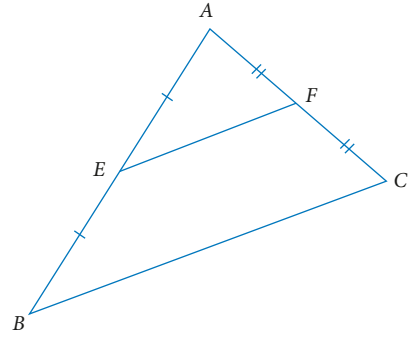
1 **Example 13** Find the value of the pronumerals. Give reasons.



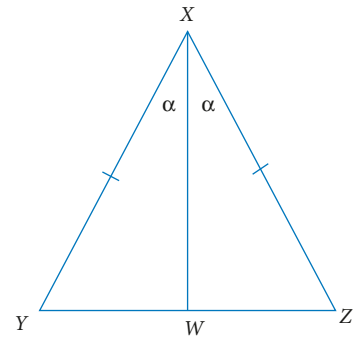


2 **Example 15** In the diagram,  $E$  and  $F$  are the midpoints of  $AB$  and  $AC$  in  $\triangle ABC$ . Prove that

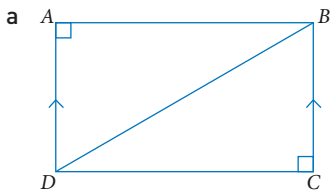
- a  $BC = 2EF$
- b  $BC \parallel EF$



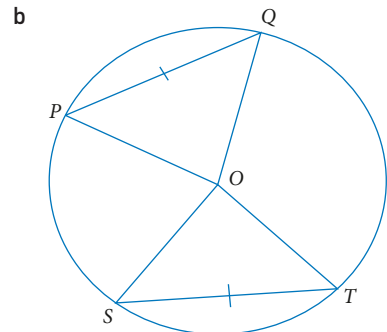
3 In the diagram,  $XY = XZ$ .  $XW$  bisects  $\angle YXZ$ . Prove that  $XW$  bisects  $YZ$ .



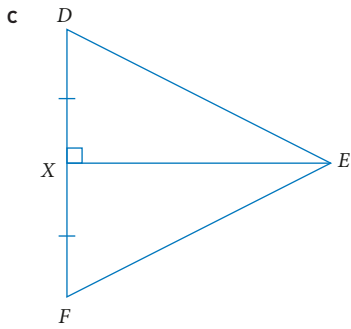
4 Recall the four triangle congruence proofs: SSS, SAS, AAS and RHS. Select the appropriate test and hence prove the following.



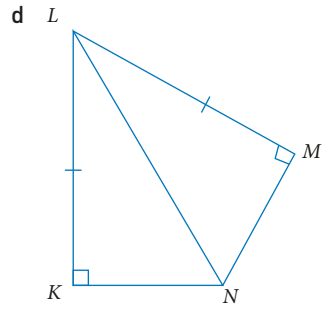
RTP:  $\triangle ADB \equiv \triangle CBD$



RTP:  $\triangle PQO \equiv \triangle TSO$



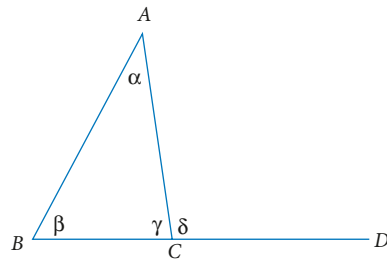
RTP:  $\triangle DXE \equiv \triangle FXE$



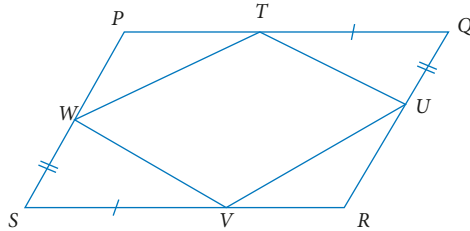
RTP:  $\triangle LKN \equiv \triangle LMN$

## Reasoning and communication

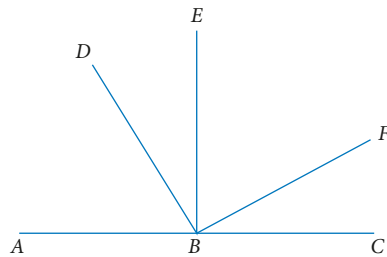
- 5 **Example 14** Prove that the exterior angle of a triangle is equal to the sum of the two interior opposite angles. Use the diagram shown.



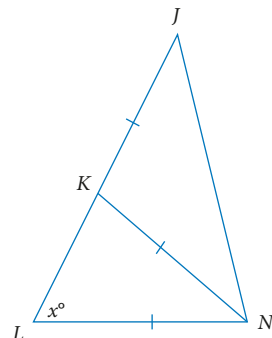
- 6 In the diagram,  $PQRS$  is a parallelogram.  $T, U, V$  and  $W$  are points lying on the sides of the parallelogram as shown.  $QU = WS$  and  $TQ = VS$ . Prove  
 a  $\triangle WSV \equiv \triangle UQT$   
 b  $TUVW$  is a parallelogram.



- 7 In the diagram,  $EB \perp AC$  and  $DB \perp BF$ . Prove that  $\angle ABF + \angle DBC = 270^\circ$ .



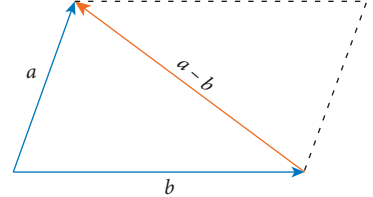
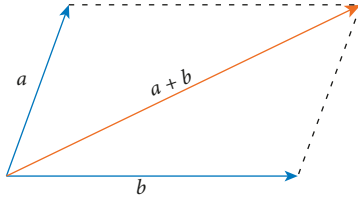
- 8 In the diagram,  $JN = JL$ ,  $JK = KN$  and  $KN = LN$ . Find the value of the pronumeral.



## 2.06 GEOMETRIC PROOFS USING VECTORS

You can use rules for vectors, including the parallelogram rule, to prove many of geometric properties of plane shapes.

Recall the parallelogram rule for vectors:



### Example 16

Use vectors to prove that the diagonals of a parallelogram bisect each other.

#### Solution

State what has to be proved.

Start by drawing a diagram and naming points.

Express the lengths of the sides as vectors.

We need to consider the two separate diagonals,  $DB$  and  $AC$ .

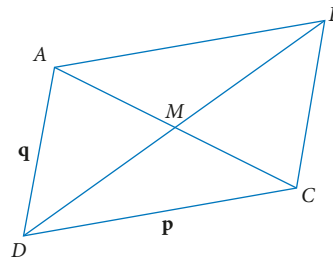
We need to show that the midpoints of  $DB$  and  $AC$ ,  $K$  and  $N$  respectively, coincide with the point  $M$ .

#### RTP

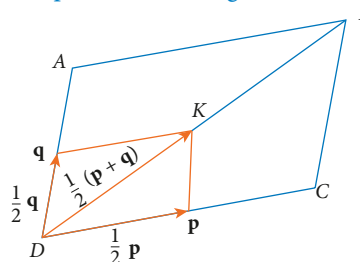
The diagonals of a parallelogram bisect each other.

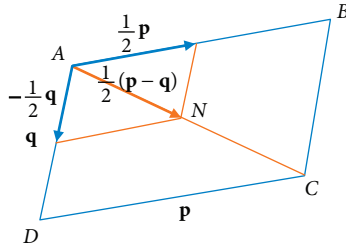
#### Proof

Draw a parallelogram with vertices of  $A, B, C, D$  and corresponding vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ . The diagonals  $AC$  and  $BD$  meet at  $M$  ( $\mathbf{m}$ ).



Let  $\overline{DC} = \mathbf{p}$  and  $\overline{DA} = \mathbf{q}$ . Let  $K(\mathbf{k})$  and  $N(\mathbf{n})$  be the midpoints of the diagonals  $DB$  and  $AC$  respectively.





We need to find the vector  $\mathbf{k}$  in terms of  $\mathbf{p}$  and  $\mathbf{q}$ . Use the parallelogram rule.

Now we need to find the vector  $\mathbf{n}$  in terms of  $\mathbf{p}$  and  $\mathbf{q}$ . Use the parallelogram rule.

We have shown that the points  $K$  and  $N$  can be represented by the same vector, so they must be the same point.

Conclude the proof.

Say you have done it.

Consider the midpoint  $K(\mathbf{k})$  of  $DB$ .

$$\overrightarrow{DB} = \overrightarrow{DC} + \overrightarrow{DA}$$

$$= \mathbf{p} + \mathbf{q}$$

So

$$\frac{1}{2}\overrightarrow{DB} = \frac{1}{2}\overrightarrow{DC} + \frac{1}{2}\overrightarrow{DA}$$

$$= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$$

$$= \frac{1}{2}(\mathbf{p} + \mathbf{q})$$

$\therefore$  If  $K(\mathbf{k})$  is the midpoint of  $DB$ , then

$$\mathbf{k} = \mathbf{d} + \frac{1}{2}(\mathbf{p} + \mathbf{q}).$$

Now consider the midpoint  $N(\mathbf{n})$  of  $AC$ .

$$\overrightarrow{AC} = \overrightarrow{DC} - \overrightarrow{DA}$$

$$= \mathbf{p} - \mathbf{q}$$

So

$$\frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\overrightarrow{DC} - \frac{1}{2}\overrightarrow{DA}$$

$$= \frac{1}{2}\mathbf{p} - \frac{1}{2}\mathbf{q}$$

$$= \frac{1}{2}(\mathbf{p} - \mathbf{q})$$

$\therefore$  If  $N(\mathbf{n})$  is the midpoint of  $AC$ , then

$$\mathbf{n} = \mathbf{a} + \frac{1}{2}(\mathbf{p} - \mathbf{q})$$

$$= (\mathbf{q} + \mathbf{d}) + \frac{1}{2}(\mathbf{p} - \mathbf{q}) \text{ since } \mathbf{q} = \mathbf{a} - \mathbf{d}$$

$$= \mathbf{d} + \frac{1}{2}(\mathbf{p} + \mathbf{q})$$

$$= \mathbf{k}$$

$\therefore$   $K$  and  $N$  are the same point, i.e., they both coincide with the point  $M$ .

$\therefore$  The diagonals of a parallelogram bisect each other.

**QED**



## ○ Example 17

Use vectors to prove that if the diagonals of a quadrilateral bisect each other, then it must be a parallelogram.

### Solution

State what is required.

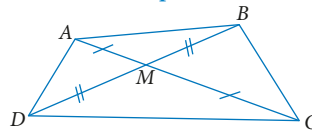
Begin the proof with a diagram and label the points and vectors.

#### RTP

If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

#### Proof

Draw the quadrilateral  $ABCD$  and label the intersection of the diagonals  $M$ . Call the position vectors of the points  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$  and  $\mathbf{m}$ .



State the assumptions in useable form.

$$AM = MC \text{ so } \mathbf{AM} = \mathbf{MC}$$

$$DM = MB \text{ so } \mathbf{MD} = \mathbf{BM}$$

Write  $\mathbf{AD}$  and  $\mathbf{BC}$  as sums of known vectors.

$$\mathbf{AD} = \mathbf{AM} + \mathbf{MD}$$

$$\mathbf{BC} = \mathbf{BM} + \mathbf{MC}$$

Use the assumptions.

$$\text{But } \mathbf{AM} = \mathbf{MC} \text{ and } \mathbf{MD} = \mathbf{BM}$$

$$\begin{aligned} \text{So } \mathbf{AD} &= \mathbf{AM} + \mathbf{MD} \\ &= \mathbf{BM} + \mathbf{MC} \\ &= \mathbf{BC} \end{aligned}$$

State the meaning of the equality.

Since  $\mathbf{AD} = \mathbf{BC}$ ,  $AD$  and  $BC$  are parallel.

There is no need to repeat the same steps with different letters.

Similarly,  $AB$  and  $DC$  are parallel.

State the overall conclusion.

Since both pairs of opposite sides are parallel,  $ABCD$  is a parallelogram.

Say you have finished.

**QED**

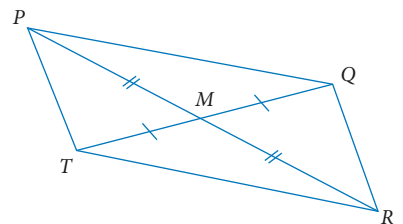
## EXERCISE 2.06

## Geometric proofs using vectors

### Concepts and techniques

- 1 **Examples 16, 17** Use vectors to prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

[Hint: In the diagram,  $M$  is the midpoint of  $PR$  and  $QT$ . Use vectors to prove that  $PQRT$  is a parallelogram.]





2 In order to prove the following statement:

*The diagonals of a parallelogram meet at right angles if and only if it is a rhombus*  
 you will need to prove two statements.

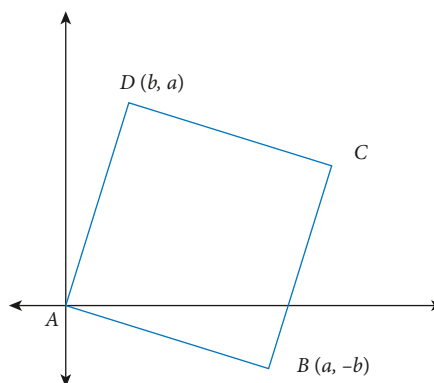
- 1 If the diagonals of a parallelogram meet at right angles, then it is a rhombus.
- 2 If it is a rhombus, then \_\_\_\_\_.

  - a Complete the statement above.
  - b Prove the first statement using vector geometry.
  - c Prove the second statement using vector geometry.
  - d What property of a rhombus does this prove?

3 Use vectors to prove that the diagonals of a square are equal in length.

Consider the diagram of the square with vectors  $\vec{AB}$  and  $\vec{AD}$  represented by the coordinates  $(a, -b)$  and  $(b, a)$  as shown.

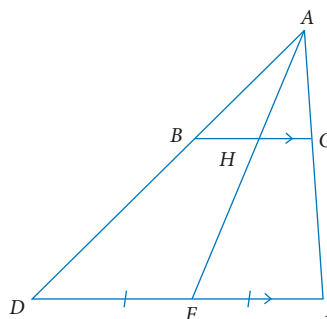
- a Find the vector  $\vec{AC}$  in terms of  $a$  and  $b$ .
- b Find the vector  $\vec{DB}$  in terms of  $a$  and  $b$ .
- c Explain why  $AC$  and  $DB$  are equal in length.



4 Use vectors to show that any line between two sides of a triangle that is parallel to the third side is bisected by the line from the midpoint of the third side to the opposite vertex.

Consider the diagram on the right.

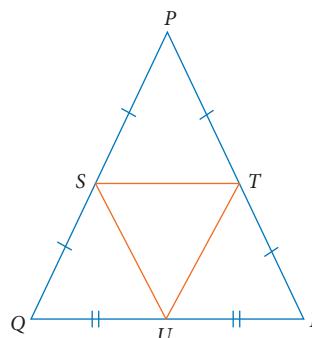
RTP:  $BH = HC$



5 Use vectors to show that the midpoints of the sides of an isosceles triangle form another isosceles triangle.

Consider the diagram on the right.

RTP:  $\triangle STU$  is isosceles.



- 6 Use vectors to show that the medians of any triangle are concurrent.

To prove this theorem, you will need to use two definitions.

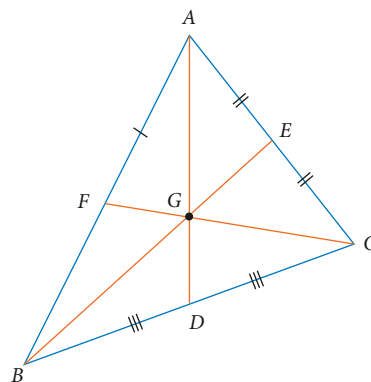
The **median** of a triangle is a line that joins the midpoint of a side to the opposite vertex.

Three or more lines are **concurrent** if they all meet at the same point.

Consider the diagram on the right.

RTP:  $FC$ ,  $BE$  and  $DA$  intersect at the point  $G$ .

[Hint: Find the point of intersection  $G$  of  $FC$  and  $BE$  and show that  $AD$  passes through  $G$ .]



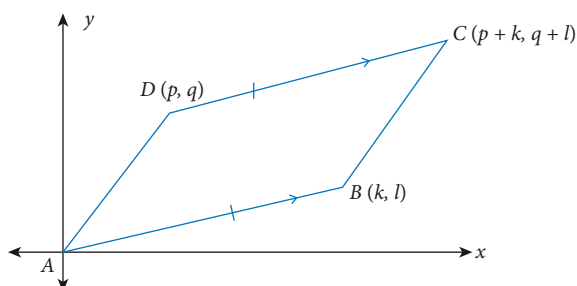
## Reasoning and communication

- 7 In order to prove the following statement:

*The line joins the midpoints of two sides of a triangle iff it is parallel to the third side and half its length*

you will need to prove two statements.

- 1 If the line joins the midpoints of two sides of a triangle, then it is parallel to the third side and half its length.
  - 2 If a line is parallel to one side of a triangle and half its length, then it \_\_\_\_\_.
    - a Complete the statement above.
    - b Prove the first statement using vector geometry.
    - c Prove the second statement using vector geometry.
- 8 Use vector geometry to prove that the midpoints of the sides of any quadrilateral join to form a parallelogram.
- 9 Use vector geometry to show that the opposite sides of a parallelogram are equal in length (Hint: Place the origin and axes so that so that the vertex  $A$  of the parallelogram  $ABCD$  is at the origin and the side  $AD$  is along the  $x$ -axis). Write the vectors  $\mathbf{AB} = (p, q)$  and  $\mathbf{AD} = (0, t)$ . Use vector geometry to show that if the opposite sides of a quadrilateral are equal in length and parallel, then it is a parallelogram.
- 10 Consider the diagram below, showing the quadrilateral  $ABCD$  represented by the coordinates shown.  $DC \parallel AB$  and  $DC = AB$ .
- a Prove that  $AD \parallel BC$ .
  - b Prove that  $AD = BC$ .
  - c Hence explain why  $ABCD$  is a parallelogram.



## 2.07

# QUANTIFIERS AND PROOF BY CONTRADICTION

You can use mathematical symbols to write mathematics in a succinct and unambiguous form. There are some mathematical symbols called *quantifiers* that are commonly used in mathematics proofs. Two of these are shown on the right.

### IMPORTANT

#### Quantifiers

$\forall$	for all
$\exists$	there exists

### Example 18

Use mathematical symbols to convert the following word sentences into mathematical statements.

- a For all rational numbers  $n$ , there exist integers  $p$  and  $q$  such that  $n = \frac{p}{q}$ .
- b For all positive real numbers  $x$  there exists a real number  $y$  such that  $2^y = x$ .

#### Solution

- a Use the quantifiers to replace 'for all' and 'there exists'. Use set notation to describe rational numbers or integers.  $\forall n \in \mathbf{Q}, \exists p, q \in \mathbf{Z}$  such that  $n = \frac{p}{q}$ .
- b We need to specify that  $x$  is real and positive.  $\forall x \in \mathbf{R}$  such that  $x > 0$ ,  
 $\exists y \in \mathbf{R}$  such that  $2^y = x$ .

In Section 2.01 you saw some examples of proof by contradiction. The next example is a well-known proof by contradiction using algebraic notation. Note the use of mathematical symbols.

### Example 19

RTP:  $\sqrt{2}$  is irrational.

#### Solution

Proof by contradiction begins by assuming the opposite of what we are trying to prove. It is also necessary to define the variables carefully.

Rearranging.

If the square of an integer is even, then the integer must be even.

#### Proof

Assume  $\sqrt{2}$  is rational, i.e., assume that  $\exists p, q \in \mathbf{Z}$  such that  $\sqrt{2} = \frac{p}{q}$ , where  $q \neq 0$  and  $p, q$  have no common factors.

Squaring both sides,  $2 = \frac{p^2}{q^2}$ .

$$\Rightarrow p^2 = 2q^2$$

$$\Rightarrow p^2 \text{ is even}$$

$$\Rightarrow p \text{ is even}$$

Equate both equations.

$$\Rightarrow p = 2k \text{ for some } k \in \mathbb{Z}$$

$$\Rightarrow p^2 = 4k^2 = 2q^2$$

$$\Rightarrow q^2 = 2k^2$$

$\Rightarrow q^2$  is even

$\Rightarrow q$  is even

We have proved that the original assumption is flawed.

But this means that both  $p$  and  $q$  have a common factor of 2.

CONTRADICTION.

$\therefore \sqrt{2}$  is irrational.

QED

*Note:* The abbreviation QED stands for *quod erat demonstrandum*, which can be translated from the Latin as *demonstrated as required*. It is commonly put at the end of a proof to show that the proof is complete.

## EXERCISE 2.07 Quantifiers and Proof by Contradiction

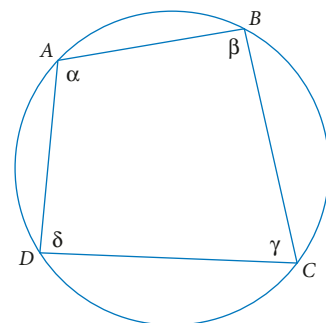


### Concepts and techniques

- Example 18** Use mathematical symbols, including quantifiers, to write each worded statement below in mathematical notation.
  - For all real numbers  $x$ , the square of  $x$  is positive or zero.
  - There exists a rational number  $w$  such that  $w$  is greater than 2 but is less than or equal to 7.
  - For all adult human beings, there exists a perfect match.
  - For all real numbers  $a$  and  $b$ , there exists a real number  $c$  such that  $c$  is between  $a$  and  $b$ .
  - For all real numbers  $x$  such that  $x \geq 0$ , there exists a real number  $y$ , where  $y \geq 0$  such that  $y = \sqrt{x}$ .
- Example 19** Follow the steps in Example 19 in order to use proof by contradiction to prove that:
  - $\sqrt{3}$  is irrational
  - $\sqrt{5}$  is irrational.
- Use proof by contradiction to prove the following theorem.

*If the four vertices of a quadrilateral lie on a circle, then the opposite angles are supplementary.*

Hint: Begin with the diagram and statement:  
Proof:  
Assume that  $A, B, C$  and  $D$  lie on a circle of centre  $O$ .  
Assume that  $\alpha + \gamma \neq 180^\circ$ .



### Reasoning and communication

Using Proof by Contradiction, prove each of the theorems below.

- The bisector of the angle between the equal sides of an isosceles triangle intersects the third side at right angles.
- The opposite sides of a parallelogram are equal in length.
- The base angles of an isosceles triangle are equal.

# CHAPTER SUMMARY

## MATHEMATICAL PROOF

# 2

- A **deductive proof** is where a number of statements are made that lead logically to a conclusion. The assumptions are called premises or **axioms**.
- An **inductive proof** is where a general conclusion is made from specific observations. In science, the general conclusion is called a **hypothesis** and it is usual to devise tests for the hypothesis.
- A **proof by contradiction** assumes that the direct opposite of a statement is true and then a logical argument is used to show that the assumption is false.
- A **counterexample** is an example that shows that a statement is not true.
- The statement *if P then Q* or *P implies Q* can be written  $P \Rightarrow Q$ .
- For the statement *if P then Q*, the **converse** is *if Q then P*  
or the **converse** of *P implies Q* is *Q implies P*.  
In symbols, the **converse** of  $P \Rightarrow Q$  is  $Q \Rightarrow P$ .
- **Equivalence**  
In words,  $P \Leftrightarrow Q$  means  
*P if and only if Q* or *P iff Q*.  
If  $P \Rightarrow Q$  and  $Q \Rightarrow P$ , then we can write  $P \Leftrightarrow Q$ .
- The negation of *P* is *not P*, written  $\neg P$ ,  $P'$  or  $\bar{P}$

### ■ Contrapositive

If we consider the statement  $P \Rightarrow Q$ , then the contrapositive statement is  $\bar{Q} \Rightarrow \bar{P}$ .

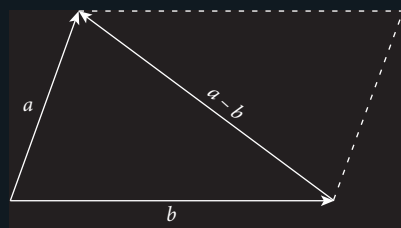
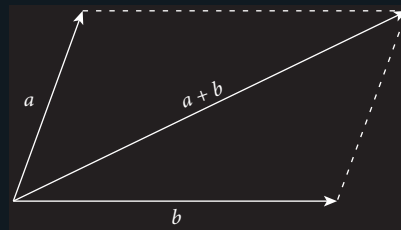
In words, the contrapositive statement of

*If P then Q* is *If not Q then not P*.

If the original statement  $P \Rightarrow Q$  is true, then the contrapositive statement  $\bar{Q} \Rightarrow \bar{P}$  is also true.

- Euclidean geometry is the geometry of two dimensions, involving lines, angles, congruence, similarity, triangles, circles and other plane shapes.

### ■ The parallelogram rule for vectors



- *RTP* = required to prove

### ■ Quantifiers

$\forall$  for all

$\exists$  there exists

- *QED* = *quod errat demonstrandum*

# 2

## CHAPTER REVIEW

### MATHEMATICAL PROOF

#### Multiple choice

- 1 **Example 11** The contrapositive of  $A \Rightarrow B$  is  
A  $B \Rightarrow A$                       B  $\bar{A} \Rightarrow \bar{B}$                       C  $A \Leftrightarrow B$                       D  $\bar{B} \Rightarrow \bar{A}$
- 2 **Example 7** Select the converse of the statement  
*If you eat poison berries, then you get sick.*  
A If you don't eat poison berries, then you don't get sick.  
B If you get sick, then you eat poison berries.  
C If you don't get sick, then you don't eat poison berries.  
D You get sick if you eat poison berries.
- 3 **Example 4** A counterexample to show that the statement '*All snakes bite*' is false would be  
A snakes don't bite  
B spiders bite  
C if you bite, then you are a snake  
D my pet snake doesn't bite
- 4 **Example 18** Choose the mathematical notation for the statement  
*For all rational numbers  $x$ , there exist integers  $y$  and  $w$  such that  $x = \frac{y}{w}$  where  $w$  is non-zero.*  
A  $\forall x \in \mathbf{Q}, \exists y, w \in \mathbf{Z}$  such that  $x = \frac{y}{w}$  where  $w \neq 0$ .  
B  $\forall x \in \mathbf{R}, \exists y, w \in \mathbf{Z}$  such that  $x = \frac{y}{w}$  where  $w \neq 0$ .  
C  $\forall x \in \mathbf{Q}, \forall y, w \in \mathbf{Z}$  such that  $x = \frac{y}{w}$  where  $w \neq 0$ .  
D  $\forall x \in \mathbf{Q}, \exists y, w \in \mathbf{N}$  such that  $x = \frac{y}{w}$  where  $w \neq 0$ .
- 5 **Example 9** The mathematical statement  $C \Leftrightarrow D$  means  
A if  $C$  then  $D$   
B  $C$  implies  $D$   
C  $C$  iff  $D$   
D  $D$  only if  $C$

#### Short answer

- 6 **Example 5** Decide whether each statement below is true or false. If false, give a counterexample.
- If the square of an integer is even, then the integer must be even.
  - If  $x^2 \geq 4$ , then  $x \geq 2$ .
  - If an animal has a pouch, then it is a marsupial.
  - If a number is prime, then it is odd.

- 7 **Example 1** Complete the deductive argument below.  
 People who practise the piano every day improve their playing.  
 Elsie practises the piano every day.  
 $\therefore$  \_\_\_\_\_.
- 8 **Example 1** Complete the inductive argument below.  
 There were floods in Queensland in 2010.  
 There were floods in Queensland in 2012.  
 There were floods in Queensland in 2014.  
 $\therefore$  \_\_\_\_\_.
- 9 **Example 7** State the converse of each statement below.
- If you are a politician, then you tell the truth.
  - If a number is positive, then the square of the number is positive.
  - If a student has failed an exam, then he/she must resit the exam.
  - If an animal is a fish, then it can swim.
- 10 **Example 11** State the contrapositive of each statement in question 9.
- 11 **Example 9** Decide which of the following true statements are equivalences. Rewrite them using *iff*.
- If you are royal, then you are often photographed.
  - If a number is composite, then it has more than two factors.
  - If a quadrilateral has diagonals that bisect each other at right angles, then it is a rhombus.
  - If an animal is a giraffe, then it eats plants.

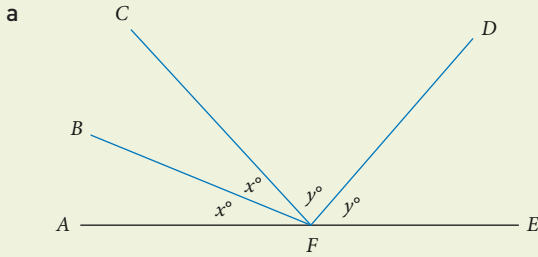
## Application

- 12 Consider the contrapositive to determine whether each statement is true.
- If a quadrilateral is a kite, then it has two pairs of adjacent sides equal.
  - If  $\exists a, b \in \mathbf{R}$  such that  $a > b$ , then  $a^2 > b^2$ .
  - If a triangle has two equal angles, then it has two equal sides.
  - If a vehicle is a bike, then it has wheels.

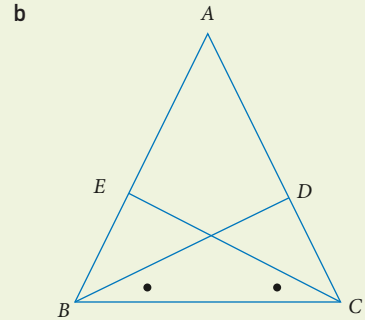


# CHAPTER REVIEW • 2

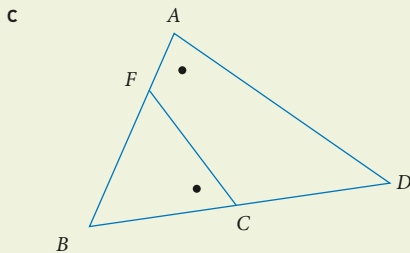
13 Use Euclidean geometry to prove each of the following.



Prove that  $\angle BFD = 90^\circ$



$AB \perp EC, AC \perp DB$   
Prove  $\triangle BEC \equiv \triangle CDB$

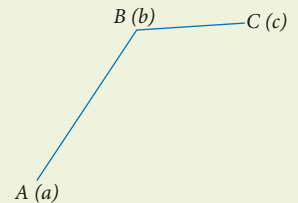


Prove  $\triangle BAD \parallel \triangle BCF$

14 The points  $A, B$  and  $C$  are represented by the vectors  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$  as shown.  $ABCD$  forms a parallelogram.

Use vectors to find:

- the fourth vertex  $D$  ( $\mathbf{d}$ ) of the parallelogram  $ABCD$ , in terms of  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$
- the midpoint  $M$  ( $\mathbf{m}$ ) of  $AC$ .



15 Use proof by contradiction to prove that  $\sqrt{7}$  is irrational.

16 Explain what is wrong with the following argument.

Ali said "If you go swimming, then you get wet."

Ben argued "No, you can get wet having a shower."

17 Decide whether the statement is 'if and only if', or 'necessary but not sufficient'.

*A trapezium has a pair of parallel opposite sides.*

18 Consider the statement:

$$\text{If } x < 2, \text{ then } \frac{1}{x^2} > \frac{1}{4}.$$

Zeni tested this with  $x = 1$  and decided that the statement was true.

Can you find a better counterexample to prove that the statement is false?

19 Write the converse of the true statement below and decide whether or not the converse is also true.

*If a median of a triangle is perpendicular to the side it intersects, then the triangle is isosceles.*

20 Write the contrapositive statement for:

$$\text{If } 0 < a < 3, \text{ then } \frac{1}{a} > \frac{1}{3}.$$

Is the statement true or false?

21 Consider the quadrilateral  $XYZQ$  as shown.

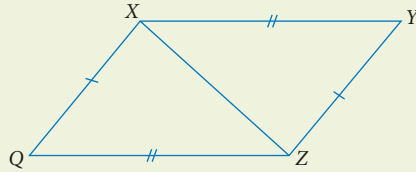
Prove that

a  $\triangle QXZ \cong \triangle YZX$

b  $XY \parallel QZ$

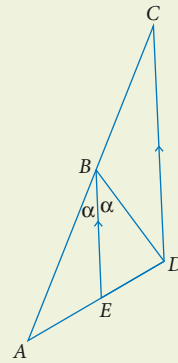
c  $XQ \parallel YZ$

d Explain why  $XYZQ$  is a parallelogram.



22 Consider the diagram on the right where  $A, B, C$  are collinear,  $BE$  bisects  $\angle ABD$  and  $BE \parallel CD$ .

Prove that  $\triangle DBC$  is isosceles.



Practice quiz